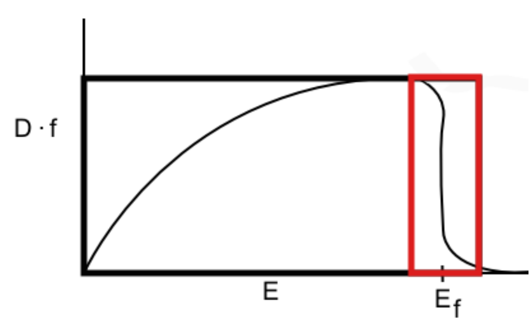
Topic 8-3: Heat Capacity
Kittel Pages: 141-147

Summary: In this video we look at two methods to solve for the electronic heat capacity. The first is a rather simple approach involving taking the area of rectangles within a graph. The second approach is more mathematically robust. Finally, we end with a discussion of why this concept is important in experiments today.

- Motivated by experiment
    - When heat capacity of metals was first measured it was much lower than expected
    - Today we will discover why the electronic heat capacity of metals is so low, not $\frac{3}{2}k_B$ per electron like a gas
- Will need to develop an expression for heat capacity of an electron gas
- Recall that the number of electrons in the system is given by $N = \int_0^\infty dE\, D(E)f(E)$
- At 0 K this gives filled states up to some maximum energy
- But as temperature increases we get smearing of our step function and we can approximate this smearing width as $2k_BT$ on either side of the Fermi energy
- This means low energy electrons will not be excited past the Fermi energy
- Thus, not all electrons will be able to contribute to the heat capacity

Approach 1: Area of Rectangles

- Begin with a ridiculously simple approach to determine the fraction of electrons that contribute to heat capacity
- For an N electron system we can invoke that there is an area contained in the rectangle below that contains electrons that can be thermally excited



- This area is $4k_BT \cdot D(E_f)$
- The total number of electrons, can be coarsely approximated by the bigger area is $E_f \cdot D(E_f)$, drawn with thick black lines
- The fraction of thermally excited electrons to the total is thus roughly $\frac{4k_BT \cdot D(E_f)}{E_f \cdot D(E_f)}$, which simplifies to $\frac{4k_BT}{E_f}$
- This ratio is useful in finding out how the electronic energy of the system changes with temperature
- Can describe thermal activation as $U = N \cdot \frac{4k_BT}{E_f} \cdot k_BT$
    - This is total number of electrons, N, times the fraction that can be thermally excited at temperature T times the energy of thermal activation
- Can rewrite E_f as k_BT_f
    - T_f is the Fermi temperature. Not a physically relevant unit, just a unit conversion trick
- Now $C = \frac{dU}{dT} = 8Nk_B \frac{T}{T_f}$
    - We have replaced the Fermi energy with the equivalent temperature and taken the derivative w/ respect to T.
- Typically the Fermi energy is 5-8 eV which makes the Fermi temperature in the 70000 K range
- This heat capacity is very consistent with experiment unlike the $\frac{3}{2}k_BT$ expression
- This helped validate quantum mechanical models of the electron as a fermion

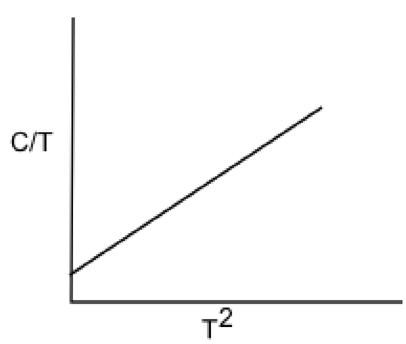
Approach 2: Electronic heat capacity - Beyond boxes!

- Taking areas of rectangles isn't a great mathematical model, even if it works
- Want a more exact calculation
- Can write the system energy as $U = \int_0^\infty dE\, D(E) E\, f(E,T)$
    - $f(E,T) = \frac{1}{e^{(E-E_f)/k_BT}+1}$
- Want an E-E_f term in the integral to simplify it

- Recall $N = \int_0^\infty dE\ D(E) f(E,T)$
  - Multiply N by E_f and take the derivative with respect to T
- $0 = \int_0^\infty dE\ D(E)\ \ E_f\ \frac{\partial f}{\partial T}$
  - Since this equation equals zero we can subtract it from our heat capacity equation
- $C - 0 = \int_0^\infty dE\ D(E)(E - E_f)\frac{\partial f}{\partial T}$
- Approximate that the density of states is constant at the Fermi energy
  - Can pull it out of the integral
- $C = D(E_f)\ \int_0^\infty dE\ (E - E_f)\frac{\partial f}{\partial T}$
  - Let $x = \frac{(E - E_f)}{k_B T}$
- $C = {k_B}^2 T\ D(E_f)\ \int_{E_f/\tau}^{\infty} dx\ \frac{x^2 e^x}{(e^x+1)^2}$
- This gives $C = \frac{1}{3}\pi^2 D(E_f) {k_B}^2 T$
  - This is linear in temperature, just like the rectangle approach!

Why is this important in experiment today

- Can approximate $C_{total} = \gamma T + \beta T^3$
  - γ is the electronic part
  - β is the phonon part
- At low temperatures, around 5 K, the electronic part is bigger than the phonon part
- Rewrite as $\frac{C}{T} = \gamma + \beta T^2$
  - Slope of the line is β, offset from the origin is γ



- From experimental γ values, one can calculate $D(E_f)$ for metals
- E-Ef df/dt is really common in solid state physics problems, where you're looking at the electrons right near the Fermi level.

Sketch the components of this function out